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Blast wave propagation in an inhomogeneous atmosphere

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The Brinkley–Kirkwood theory (1947) is modified to determine the law of propagation of a blast wave in an arbitrary inhomogeneous medium for spherically and cylindrically symmetric cases. The shock path is obtained in terms of a simple quadrature. The numerical results for the shock path and the entire flow region behind the shock, propagating in an exponential atmosphere, show excellent agreement with the exact numerical solution.

1. Introduction

The nonlinear partial differential equations governing gas flows do not generally admit exact solutions. This is particularly so for flows involving shock discontinuities. Some similarity solutions have, however, been obtained in a closed form under very special conditions. A few approximate methods which give flow conditions just behind the shock as it propagates have also been proposed but the flow in the entire region behind the shock is not determined. The two methods in this category, which have been widely used, are the Chester, Chisnell and Whitham (CCW)† method and the Brinkley–Kirkwood (BK) theory. Recently, Laumbach & Probstein (1969) have suggested another approximate method for studying the propagation of a blast wave in a medium of an exponentially decreasing and increasing density. Their technique is based on the basic assumption, Chernyi (1959), that the mass in the blast wave is almost entirely concentrated on the shock-surface heading the blast. They have employed an integral method of the type discussed by Hayes & Probstein (1966). Their numerical results agree very well with the exact numerical solution in the medium with exponentially increasing density and also up to a few scale heights for the medium with exponentially decreasing density. They have also considered the shape of the shock under the local radiality condition.

The purpose of the present paper is to give a modification of the BK theory, Brinkley & Kirkwood (1947), to obtain the solution of a blast wave problem in an inhomogeneous medium under the well-known assumption that the total energy of the blast is constant during its propagation. The BK theory has been widely used in astrophysical contexts, Sachdev (1968, 1970), Nadezhin & Frank-Kamenetskii (1965). It is known to give good results when the flow becomes self-similar in a region far away from the source. In the present paper

† See Chester (1954), Chisnell (1955) and Whitham (1958).

we first modify this theory for the constant energy solution, evaluating the similarity parameter $\nu(\gamma)$ in this theory from the exact similarity solution of Taylor (1950) for the blast wave propagation in a uniform medium. We make the assumption that this similarity parameter will serve in the inhomogeneous case too. This is the only assumption in the derivation of the shock propagation law; otherwise the BK theory makes exact use of the governing differential equations. This assumption indeed gives good agreement with the exact numerical solution. Thus we obtain a technique which does not suffer from the rather restrictive assumption of Laumbach & Probstein, that the entire mass of the blast is concentrated at the shock, which has been used at several points in the derivation of the shock law. On the other hand, we use the expansion about the shock suggested by Laumbach & Probstein to get flow parameters in the region behind the shock *after* the shock law is known from the BK theory. We find an excellent agreement with the exact numerical results of Troutman & Davis (1965). Our modification of the BK theory may be compared with the modification of the CCW method by Hayes (1968).

2. Theory

The BK theory makes use of the equations of motion and continuity in a hybrid Lagrangian–Eulerian form specialized at the shock front. A third equation is obtained by differentiating the momentum equation from the Rankine–Hugoniot conditions along the shock path. The fourth relation between the partial derivatives at the shock is obtained by imposing a similarity restraint on the energy–time curve of the shock wave and considering the dissipation of shock energy. These four relations are solved simultaneously to obtain two ordinary differential equations giving the variation of shock energy and shock strength with the distance traversed by the shock. We omit the details, for which reference may be made to Sachdev (1967) and Kogure & Osaki (1962). We further assume here that the energy of the shock E is constant so that in the BK theory we get only one equation, giving the pressure behind a *strong* shock in an inhomogeneous medium,

$$\frac{1}{p^2}\frac{dp}{dR}+\left[\frac{2\alpha(2\gamma^2-\gamma+1)}{(\gamma+1)(5\gamma-1)}\frac{1}{R}-\frac{2\gamma}{5\gamma-1}\frac{1}{\rho_0}\frac{d\rho_0}{dR}\right]\frac{1}{p}=\frac{-4\alpha\pi\nu}{5\gamma-1}\frac{R^\alpha}{E}. \tag{2.1}$$

Here p, ρ_0 and R are the pressure behind the shock, undisturbed density ahead of the shock and shock radius, measured from the source, respectively, and γ is the ratio of specific heats, $\gamma = c_p/c_v$. The similarity parameter ν is assumed to be a function of γ and $\alpha = 2, 1$ for the spherically and cylindrically symmetric cases respectively. We note that if we adopt Schatzman's (1949) assumption as to the particle path after the particle crosses a strong shock we get the shock energy to be exactly constant. We consider in detail the case $\alpha = 2$ in order to compare our results with those of Laumbach & Probstein (1969). After reducing R by a characteristic length Δ, which in the exponential medium appears in the density law

$$\rho = \rho_\beta \exp(-r_0/\Delta), \tag{2.2}$$

r_0 being the Lagrangian co-ordinate, reducing t by $(E/4\pi\rho_\beta\Delta^5)^{\frac{1}{2}}$ and ρ by ρ_β (a constant), we integrate equation (2.1) to obtain

$$\dot{R}^2 = \frac{(\gamma+1)(5\gamma-1)}{4\nu(\gamma)}\rho_0^{(1-3\gamma)/(5\gamma-1)} R^{[-4(2\gamma^2-\gamma+1)]/[(\gamma+1)(5\gamma-1)]} \times \left[\int_0^R R^{[2(\gamma^2+6\gamma-3)]/[(5\gamma-1)(\gamma+1)]}\rho_0^{2\gamma/(5\gamma-1)}\,dR\right]^{-1}. \tag{2.3}$$

To obtain the dependence of ν on γ we compare the solution of (2.1) for the uniform medium,

$$R^5t^{-2} = \frac{25}{64\pi}\frac{7\gamma^2+16\gamma-7}{\nu(\gamma)}\frac{E}{\rho_0}, \tag{2.4}$$

with the exact solution of Taylor (1950),

$$R^5t^{-2} = (1/k)(E/\rho_0), \tag{2.5}$$

where k is tabulated for different values of γ in the paper by Taylor (1950) and is also given in our table 1 below. By comparing (2.4) and (2.5) we find that $\nu(\gamma) = (25k/64\pi)(7\gamma^2+16\gamma-7)$. Thus ν can be found for each γ by suitable interpolation; we use the same value of ν for the inhomogeneous case.

γ	1·2	1·3	1·4	1·667
k	1·727	1·167	0·856	0·487

TABLE 1. Values of k versus γ (Taylor 1950) for the spherically symmetric case

The formula (2.3) for the propagation of a spherical blast wave in an inhomogeneous medium is quite general and involves only a simple quadrature.

3. Results for an exponential medium

To compare the numerical results as obtained by the formula (2.3) with the available exact numerical solution of Troutman & Davis (1965), we write the shock laws for the media with exponentially decreasing and increasing density, with upper sign in the following equation corresponding to the former and the lower to the latter:

$$\dot{R}^2 = \frac{\gamma(\gamma+1)}{2\nu(\gamma)}\left(\frac{5\gamma-1}{2\gamma}\right)^{[-2(\gamma^2+6\gamma-3)]/[(\gamma+1)(5\gamma-1)]} \times \exp\left(\pm\frac{(3\gamma-1)R}{5\gamma-1}\right) R^{[-4(2\gamma^2-\gamma+1)]/[(\gamma+1)(5\gamma-1)]} \times \left[\int_0^{2\gamma R/(5\gamma-1)} z^{[2(\gamma^2+6\gamma-3)]/[(\gamma+1)(5\gamma-1)]}\,e^{\mp z}\,dz\right]^{-1}. \tag{3.1}$$

To get the flow in the region behind the shock we follow Laumbach & Probstein (1969). Using the shock law (3.1) and retaining terms up to second order in the expansion for the Eulerian co-ordinate r about the shock front we have

$$r = R\left[1+\frac{\gamma-1}{\gamma+1}(\xi-1)+\frac{R}{2}\frac{\partial^2 r}{\partial r_0^2}\bigg|_R^{\mp}(\xi-1)^2\right], \tag{3.2}$$

$$\frac{p}{p_s} = 1-R^3\int_1^\xi e^{\mp R(\bar{\xi}-1)}\frac{\bar{\xi}^2}{r^2}\left(\frac{\gamma+1}{2}\frac{1}{\dot{R}^2}\frac{\partial^2 r}{\partial t^2}\right)^{\mp}d\bar{\xi}, \tag{3.3}$$

$$\rho/\rho_s = (p/p_s)^{1/\gamma}, \tag{3.4}$$

where
$$\frac{\gamma+1}{2}\frac{1}{\dot{R}^2}\frac{\partial^2 r}{\partial t^2}\bigg|^{\mp} = f^{\mp}(R)+\frac{\gamma+1}{2}\left[1-(\xi-1)Rf^{\mp}(R)\right]\frac{\partial^2 r}{\partial r_0^2}\bigg|_R^{\mp}, \tag{3.5}$$

$$\frac{\partial^2 r}{\partial r_0^2}\bigg|_R^{-} = \frac{2(\gamma-1)}{(\gamma+1)^2}\left[-\frac{\gamma+1}{2(5\gamma-1)}+\frac{2(4\gamma-3)}{(5\gamma-1)R}-\left(\frac{3\gamma}{5\gamma-1}\right)\left(\frac{2\gamma R}{5\gamma-1}\right)^{[2(\gamma^2+6\gamma-3)]/[(\gamma+1)(5\gamma-1)]}\right.$$
$$\left.\times\exp\left(\frac{-2\gamma}{5\gamma-1}R\right)\times\left\{\int_0^{2\gamma R/(5\gamma-1)} e^{-z}z^{[2(\gamma^2+6\gamma-3)]/[(\gamma+1)(5\gamma-1)]}dz\right\}^{-1}\right], \tag{3.6}$$

$$\frac{\partial^2 r}{\partial r_0^2}\bigg|_R^{+} = \frac{2(\gamma-1)}{(\gamma+1)^2}\left[\frac{\gamma+1}{2(5\gamma-1)}+\frac{2(4\gamma-3)}{(5\gamma-1)}\frac{1}{R}-\frac{3\gamma}{5\gamma-1}\left(\frac{2\gamma R}{5\gamma-1}\right)^{[2(\gamma^2+6\gamma-3)]/[(\gamma+1)(5\gamma-1)]}\right.$$
$$\left.\times\exp\left(\frac{2\gamma R}{5\gamma-1}\right)\left\{\int_0^{2\gamma R/(5\gamma-1)} e^{z}z^{[2(\gamma^2+6\gamma-3)]/[(\gamma+1)(5\gamma-1)]}\right\}^{-1}\right], \tag{3.7}$$

$$f^{-}(R) = \frac{1}{2}\frac{3\gamma-1}{5\gamma-1}-\frac{2(2\gamma^2-\gamma+1)}{(\gamma+1)(5\gamma-1)}\frac{1}{R}-\frac{\gamma}{5\gamma-1}\left(\frac{2\gamma R}{5\gamma-1}\right)^{[2(\gamma^2+6\gamma-3)]/[(5\gamma-1)(\gamma+1)]}$$
$$\times\exp\left(\frac{-2\gamma R}{5\gamma-1}\right)\left[\int_0^{2\gamma R/(5\gamma-1)} e^{-z}z^{[2(\gamma^2+6\gamma-3)]/[(5\gamma-1)(\gamma+1)]}dz\right]^{-1}, \tag{3.8}$$

$$f^{+}(R) = -\frac{(3\gamma-1)}{2(5\gamma-1)}-\frac{2(2\gamma^2-\gamma+1)}{(\gamma+1)(5\gamma-1)}\frac{1}{R}-\frac{\gamma}{5\gamma-1}\left(\frac{2\gamma R}{5\gamma-1}\right)^{[2(\gamma^2+6\gamma-3)]/[(\gamma+1)(5\gamma-1)]}$$
$$\times\exp\left(\frac{2\gamma R}{5\gamma-1}\right)\left[\int_0^{2\gamma R/(5\gamma-1)} e^{z}z^{[2(\gamma^2+6\gamma-3)]/[(5\gamma-1)(\gamma+1)]}dz\right]^{-1}. \tag{3.9}$$

In the above, $-$ and $+$ correspond to quantities pertaining to the decreasing and increasing density medium respectively, r_0 is the Lagrangian co-ordinate and $\xi = r_0/R$. The subscript s denotes conditions just behind the shock. We omit the details of the deduction of the above formulae which may be easily obtained by referring to the paper by Laumbach & Probstein (1969).

We have compared the numerical values of $\dot{R}$ and R as well as the pressure and density distributions in the entire region behind the shock for both increasing and decreasing density media with those of Probstein & Laumbach (1969, figures 3, 4, 6 and 7). The agreement for the increasing density medium is excellent. For the medium with the decreasing density, it is good only up to $R \sim 6\Delta$ after which $\dot{R}$ increases much more rapidly in our case (figure 1). Since the exact numerical results are not available for this range of R it is difficult to make a definite conclusion. However, the approximation in the method of Laumbach & Probstein, that the entire mass behind the shock is concentrated on the shock surface itself, improves as the shock propagates in the downward direction while it grows worse in the upward direction with the density decreasing exponentially. This might explain the divergence of our results from those of Laumbach & Probstein. We have not given the comparative results corresponding to figures 4, 6 and 7 of their paper since the results agree so well that they cannot be distinguished on the graph.

The motion assumes a self-similar character when the shock has moved far from the source, so that there is no longer any dependence on the energy of the blast, and the shock strength changes owing to the local non-uniformities.

Sachdev (1970) has given a comparative study of results obtained by the method of characteristics, the CCW method and the BK theory for different density inhomogeneities. The general conclusion reached was that the CCW method and the BK theory have a more or less comparable accuracy, though the BK theory gave a little better results in some cases. We note that the comparative study of the CCW method and BK theory by Kaplan (1967) is erroneous because of the omission of a factor $\sqrt{2}$ in his expression for shock law according to the CCW method. The agreement between the two theories is much better after this correction has been introduced.

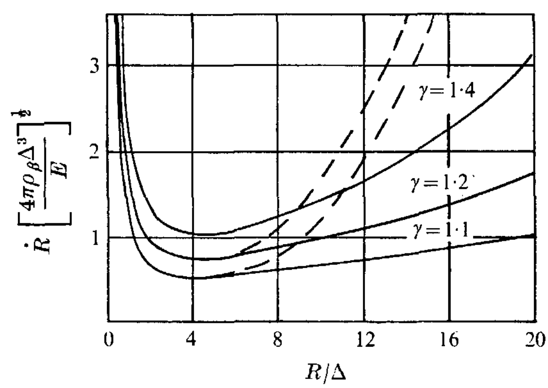

FIGURE 1. Shock velocity in upward direction as a function of shock position. ——, Laumbach & Probstein (1969); – – –, modified BK theory.

4. Concluding remarks

We have given a modified form of the BK theory to study the propagation of a blast wave in an (arbitrary) inhomogeneous medium. The shock path for a general density distribution in the undisturbed medium and for the spherically and cylindrically symmetric cases is obtained in terms of a simple quadrature. The numerical results obtained by this method agree remarkably with the exact numerical solutions. The flow field behind the shock is obtained following an earlier paper by Laumbach & Probstein *after* the shock path has been obtained by the modified BK theory. These again show good agreement with exact numerical solution.

We hope to give, in a future communication, an extension of these results to the consideration of effects of explosions at large distances, where the counter-pressure is important, and also to the determination of the shape of the shock during the course of its propagation.

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